Lecture 12

For now we're still following Onishchik-Vinberg (OV1994).

<u>Compactness theorem</u> G complex semisimple $P \subset G$ parabolic.
Then G/P is compact.

<u>Pf</u>. Can take $P = P_S$. Also, enough to show for B since $G/B \to G/P_S$ surj.

constr involves algebra.

Sketched last time: $\exists$ projective variety ($X \subset \mathbb{P}(\mathbb{C}^N)$ polynom eqn) w/ G-action s.t.

- B is the stabilizer of <u>some</u> point ($G/B = G \cdot x_0$)
- The stabilizer of every point is solvable.

Then maximality of $B \Rightarrow G \cdot x_0$ minimal dim $\Rightarrow$ closed in $\mathbb{P}(\mathbb{C}^N)$
$\Rightarrow$ compact. $\square$

<u>Alternate approach</u>. The compact real form.

If m is a Lie alg over $\mathbb{R}$, then the vec space $m \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ has a natural $\mathbb{C}$ Lie alg str. Complex linearity determines $[,]$.

$m \otimes_{\mathbb{R}} \mathbb{C} \underset{\mathbb{R} \text{ vec sp.}}{\cong} m \oplus (im)$ $\quad$ $m \otimes_{\mathbb{R}} \mathbb{C}$ denoted $m^{\mathbb{C}}$ complexification.

Conversely, a real form of $\mathfrak{g}$ (a Lie alg over $\mathbb{C}$) is a $\mathbb{R}$-subalg m s.t. the inclusion $m \hookrightarrow \mathfrak{g}$ extends to a $\mathbb{C}$ Lie alg iso $m^{\mathbb{C}} \xrightarrow{\sim} \mathfrak{g}$, or equiv, $\mathfrak{g} \underset{\mathbb{R} \text{ vec sp.}}{\cong} m \oplus im$

$\mathfrak{g}$ (semi)simple $\Rightarrow$ every real form is (semi) simple.

Two special: ① Real form w/ pos def Killing form $\to$ SPLIT.

② Real form w/ neg def Killing form $\longrightarrow$ COMPACT
$\Leftrightarrow \mathfrak{k} = \mathrm{Lie}(K)$ where $K \subset G$ is compact.
(in fact maximal)

e.g. $\mathfrak{sl}_n\mathbb{C}$ $\nearrow$ split $\mathfrak{sl}_n\mathbb{R} = \{X \text{ real}, \ \mathrm{tr}\, X = 0\}$
$\searrow$ compact $\mathfrak{su}(n) = \{X \text{ cplx} \mid X + \bar{X}^T = 0, \ \mathrm{tr}\, X = 0\}$

Fact from Lie theory: $\mathfrak{g} \supset \mathfrak{k}$ compact real form ($\mathfrak{g} \underset{\mathbb{R}\text{ vec}}{\cong} \mathfrak{k} \oplus (i\mathfrak{k})$)
Then $\exists$ solvable $\mathbb{R}$-subalg $\mathfrak{s} \subset \mathfrak{g}$ s.t. $\mathfrak{g} \underset{\mathbb{R}\text{vec}}{\cong} \mathfrak{k} \oplus \mathfrak{s}$

This is part of the so-called Iwasawa decomposition.

Ex. $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ $\mathfrak{k} = \mathfrak{su}(n)$ $\mathfrak{s}$ = upper tri complex matrices w/ real entries on diag

$$X = \begin{pmatrix} d_1 & & U \\ & \ddots & \\ L & & d_n \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} \mathrm{Im}\, d_1 & & \bar{L}^T \\ & \ddots & \\ L & & \mathrm{Im}\, d_n \end{pmatrix}}_{\in\, \mathfrak{k}} + \underbrace{\begin{pmatrix} \mathrm{Re}\, d_1 & & U - \bar{L}^T \\ & \ddots & \\ 0 & & \mathrm{Re}\, d_n \end{pmatrix}}_{\in\, \mathfrak{s}}$$

Warning. $\mathfrak{k}, \mathfrak{s}$ don't commute!
Also, $\mathfrak{s}$ contains a real form of a Cartan which is enough to be self-normalizing. $\exists$ subgrp $S \subset G$

Pf of cptness: Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, conjugating so $\mathfrak{s} \subset \mathfrak{b}$, $\mathfrak{b} + \mathfrak{k} = \mathfrak{g}$
Then consider $G \to G/B$. Then $\pi|_K$ has $(d\pi|_K)_e$ surjective ($\Leftrightarrow \mathfrak{k} + \mathfrak{b} = \mathfrak{g}$) so image of π contains nbd of eB.
The map is equivt under left G-action, so image $\pi(K)$ open.
K compact so $\pi(K)$ closed. G/B connected $\Rightarrow \pi(K) = G/B$.
Thus G/B is the continuous image of a compact set.
$\square$

Explicit models.

Type A_n. G/B = full flag variety of $\mathbb{C}^{n+1}$ G/P_{max} = Grassmannian.

Type $B_n = so_{2n+1}\mathbb{C}$ G/B = isotropic flags in $(\mathbb{C}^{2n+1}, b)$

Type C_n $sp(2n, \mathbb{C})$ G/B = isotropic flags in $(\mathbb{C}^{2n}, \omega)$

Type D_n G/B = isotropic flags in $(\mathbb{C}^{2n}, b)$ of form

$$\begin{array}{l} F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n^+ \\ \qquad\qquad\qquad\quad \cap \\ \qquad\qquad\qquad\quad F_n^- \end{array}$$

where $F_n^+ \cap F_n^0$ has even dim, $F_n^- \cap F_n^0$ has odd dim.

Topology.

Thm. G complex semisimple, connected. Then G/B is simply connected and has a CW complex structure with all cells of even dimension.
The number of cells is $|G(w)|$. Further, there is one 0-cell and $|\Delta| = rk(G)$ 2-cells.

In proving this we'll use more properties of the Iwasawa decomposition. REF: Helgason.